

Thm: $X: \Omega \rightarrow \mathbb{R}$; $Y: \Omega \rightarrow \mathbb{R}$ be independent Random Variables. Then $E(XY) = E(X) \cdot E(Y)$

Proof:

$$E(XY) = \sum c_k \cdot P(XY = c_k)$$

$$= \sum c_k \sum_{a_i \cdot b_j = c_k} P(X = a_i \cap Y = b_j)$$

$$\{XY = c_k\} = \bigcup_{a_i \cdot b_j = c_k} \{X = a_i\} \cap \{Y = b_j\}$$

$\{X = a_i\} \cap \{Y = b_j\}$ are disjoint

$$\therefore P\{XY = c_k\} = \sum_{a_i \cdot b_j = c_k} P(X = a_i \cap Y = b_j)$$

Independence \Rightarrow

$$E(XY) = \sum c_k \sum_{a_i \cdot b_j = c_k} P(X = a_i) \cdot P(Y = b_j)$$

$$E(XY) = \sum_k \sum_{a_i, b_j = c_k} c_k P(X=a_i) P(Y=b_j)$$

$$= \sum_k \sum_{a_i, b_j = c_k} a_i P(X=a_i) b_j P(Y=b_j)$$

$$E(X) = \sum_i a_i P(X=a_i); E(Y) = \sum_j b_j P(Y=b_j)$$

$$E(X)E(Y) = \sum_{i,j} a_i P(X=a_i) \cdot b_j P(Y=b_j)$$

$$= \sum_k \sum_{a_i, b_j = c_k} a_i P(X=a_i) \cdot b_j P(Y=b_j)$$

Since $\ast \subset \ast\ast$ are the same.

$$E(XY) = E(X) \cdot E(Y)$$

QED

How do we show if X, Y, Z
are mutually independent
then $E(XYZ) = E(X)E(Y)E(Z)$

Like to say X, Y, Z are independent
 $E(XYZ) = E(XY)E(Z) = E(X)E(Y)E(Z)$

Case of
3 variables
for expected
value.

In above I have considered XY as
one variable and Z as other.

Similarly, we have proven that
the rule for 2 independent RV

$$E(XY) = E(X)E(Y)$$

$$\text{implies } \sigma^2(X+Y) = \sigma^2(X) + \sigma^2(Y)$$

How about $\sigma^2(X+Y+Z)$. If we
know $X+Y$ and Z are independent

$$\text{then } \sigma^2((X+Y)+Z) = \sigma^2(X+Y) + \sigma^2(Z)$$

$$= \sigma^2(X) + \sigma^2(Y) + \sigma^2(Z)$$

when X, Y, Z are mutually !!
independent.

There are analogous results for
 $E(X_1 X_2 \dots X_n) = E(X_1) E(X_2) \dots E(X_n)$
 $\sigma^2(X_1 + X_2 + \dots + X_n) = \sigma^2(X_1) + \dots + \sigma^2(X_n)$ under
independence assumptions

To take care of both cases for $n=3$
let's prove that if X_1, X_2, X_3 are
mutually independent then

$f(X_1, X_2)$ and X_3 are indep.

ex: $f(X_1, X_2) = X_1 + X_2$ ← the two cases we
 $g(X_1, X_2) = X_1 X_2$ consider

to show $P(X_3 = d \cap f(X_1, X_2) = c)$
 $= P(X_3 = d) P(f(X_1, X_2) = c)$

Lemma A

For $f(X_1, X_2) = X_1 + X_2$ Lemma A
will show $X_1 + X_2$ & X_3 are independent

For $f(X_1, X_2) = X_1 X_2$ Lemma A
will show $X_1 X_2$ & X_3 are independent

The next page has the proof
of Lemma A

$$P(X_3=d) P(F(X_1, X_2)=c) \stackrel{?}{=} P(X_3=d \cap \{F(X_1, X_2)=c\})$$

But the right hand side above =

$$P(\{X_3=d\} \cap \bigcup_{F(a_i, b_j)=c} \{X_1=a_i\} \cap \{X_2=b_j\})$$

right hand
side

$$= P(\bigcup_{F(a_i, b_j)=c} \{X_3=d\} \cap \{X_1=a_i\} \cap \{X_2=b_j\})$$

by independence of 3 variables X_1, X_2, X_3

$$= \sum_{F(a_i, b_j)=c} P(X_3=d) P(X_1=a_i) P(X_2=b_j) \quad *$$

$$F(a_i, b_j)=c$$

The top left hand side =

$$= P(X_3=d) \cdot P(\bigcup_{F(a_i, b_j)=c} \{X_1=a_i\} \cap \{X_2=b_j\})$$

as before

$$= P(\{X_3=d\}) \cdot \sum_{F(a_i, b_j)=c} P(\{X_1=a_i\} \cap \{X_2=b_j\})$$

and by pairwise independence of X_1 and X_2

$$= P(\{X_3=d\}) \cdot \sum_{F(a_i, b_j)=c} P(\{X_1=a_i\}) \cdot P(\{X_2=b_j\})$$

$$= * \text{ by distribution of multiplication}$$

Lemma A shows that
if x, y, z are mutually independent
then

$$E(xyz) = E(x)E(y)E(z)$$

$$\sigma^2(x+y+z) = \sigma^2(x) + \sigma^2(y) + \sigma^2(z)$$

The case for general n will
be a project for the grad
students